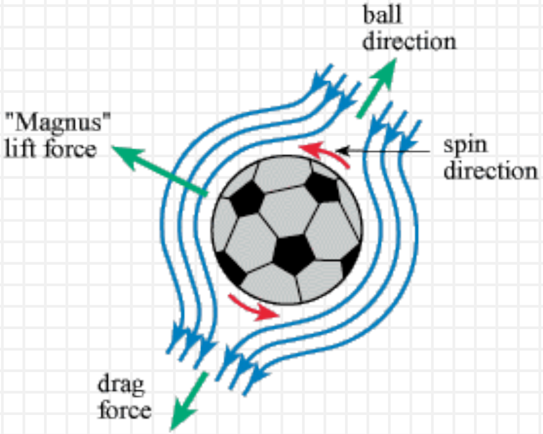
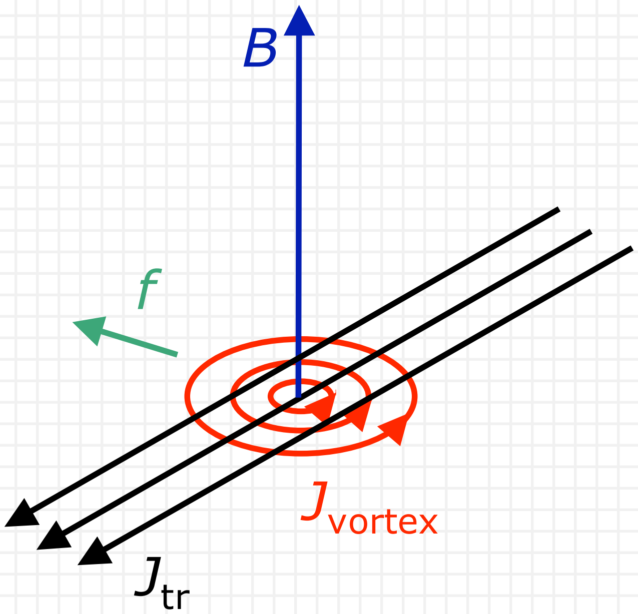
Last lecture (#4):

We completed the discussion of the *B*-*T* phase diagram of type-I and type-II superconductors. In contrast to type-I, the type-II state has finite resistance unless vortices are pinned by defects.





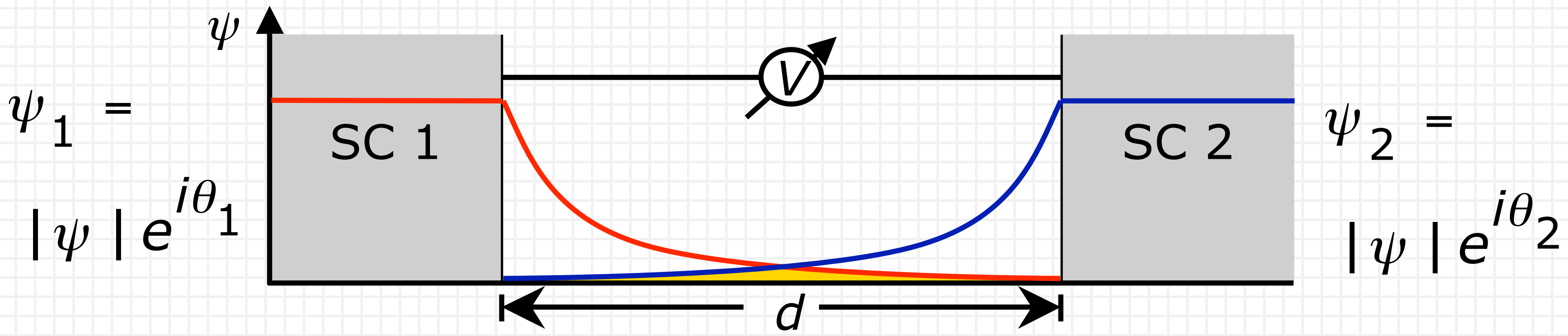
- Homogeneous state → Meissner effect → type-I sc
- Inhomogeneities at isolated points → vortices → type-II sc
- Inhomogeneities at weak links → Josephson effect, SQUIDs

Lecture 5:

- Weak links and the Josephson phase relation
- Josephson critical current
- DC and AC Josephson effect with voltage source (current source given in an appendix)
- Gauge-invariant phase
- Quantum interference for weak links
- The DC SQUID
- Applications of SQUIDS
- Other applications of Josephson phenomena: frequency mixers and voltage standards

- Literature: Waldram chs 6 & 18

Weak Links



- Bulk superconductors 1 & 2 are separated by a very thin region of normal metal or insulator.

- A simple model of a 1D superconducting weak link, $d \ll \xi$, and strong link, $d \gg \xi$, is given in the appendix. Here we consider a more general phenomenological approach

- Phase difference $\varphi = \theta_1 - \theta_2$ evolves as (lecture 2)

$$\partial\varphi / \partial t = 2eV / \hbar$$

- In GL free energy density and in expression for current we replace

$$-\nabla\psi \quad \text{by} \quad \frac{\psi_1 - \psi_2}{d} \quad \propto \quad exp(i\varphi) - 1$$

- The current through the link is then of the form

$$\boxed{I = I_J \sin\varphi}$$

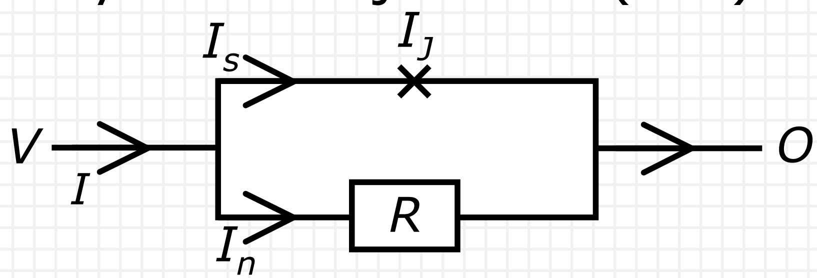
In a weak link (*i.e.*, $d << \xi$ for 1D sc link) the current is periodic in φ with period 2π.

- The current I is consistent with a free energy term of the form $\Delta F = -F_0 \cos\varphi$, where $F_0 = \hbar I_J/(2e)$. Proof:

Power $= \partial \Delta F / \partial t = F_0 \sin\varphi \, \partial\varphi / \partial t = IV$ as required.

Voltage-Biased Josephson Weak Links
The Josephson Current-Phase Equation

Consider the resistively shunted junction (RSJ):



The total current with bias voltage *V* is

$$I = I_s + I_n = I_J \sin\varphi + \frac{V}{R}$$

Since $V = (\hbar/2e)\partial\varphi/\partial t$, we can rewrite I in terms of the phase φ alone

$$I = I_J \sin\varphi + \frac{\hbar}{2eR}\frac{\partial\varphi}{\partial t}$$

This is a strange circuit equation unlike any known in conventional circuit theory and it leads to remarkable *I*-*V* characteristics. I_J is known as the Josephson critical current of the weak link and is a constant that depends on the microscopic details of the junctions. Typical values of I_J are in the range 10^{-6} A to 10^{-2} A.

The Josephson Current-Voltage Relation

From the phase-voltage relation $V = (\hbar/2e)\partial\varphi/\partial t$, we can write φ as an integral over $V(t)$

$$\varphi = \varphi_0 + \frac{2e}{\hbar}\int_0^t V(t)dt$$

where φ_0 is a constant. Thus, the current-voltage form of the Josephson equation becomes

$$I = I_J \sin\left(\varphi_0 + \frac{2e}{\hbar}\int_0^t V(t)dt\right) + \frac{V}{R}$$

Consider first the case $V = 0$. Then $I_n = 0$ and

$$I = I_S = I_J \sin\varphi_0$$

Current flows without an applied voltage, i.e., I_s is indeed a supercurrent flowing through the weak link. This is the DC-Josephson effect.

AC-Josephson Effect

The main surprise comes when we apply a finite voltage. Consider first a DC voltage V. This leads to a time-dependent phase

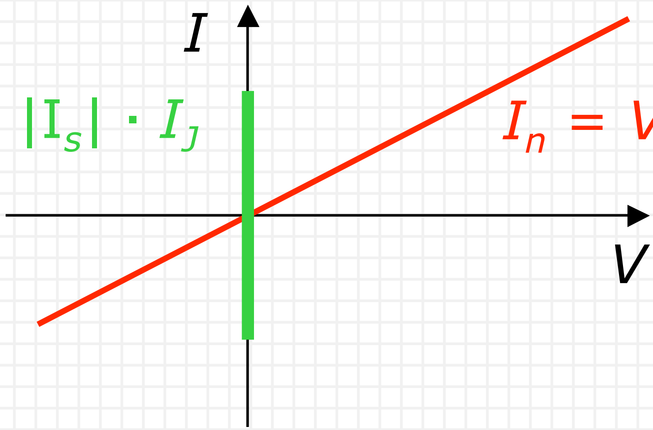
$$\varphi = \varphi_0 + \frac{2eV}{\hbar} t$$

and thus to an oscillatory component in the current

$$I = I_J \sin(\varphi_0 + \omega_J t) + \frac{V}{R}, \quad \text{where} \quad \omega_J = \frac{2eV}{\hbar} = 2\pi V / \phi_0$$

$f_J = \frac{\omega_J}{2\pi} = \frac{V}{\phi_0} = (4.8359\ldots \; x \; 10^8 \text{Hz}/1\mu\text{V}) \; V$ is the Josephson frequency.

Remarkably, a DC applied voltage drives an oscillating DC super-current at a frequency that is $(1/\phi_0)$ per unit of voltage applied. This is the AC-Josephson Effect. The DC I-V characteristic of a RSJ weak link is given on the right.



Combined DC and AC Applied Voltages

Now include both a DC and an AC voltage $V = V_0 + V_{RF}\cos(\omega_{RF} t)$ so that

$$\varphi = \varphi_0 + \omega_{J0}\, t + \frac{\omega_{JRF}}{\omega_{RF}} \sin(\omega_{RF}\, t)$$

where $\omega_{J0} = 2eV_0/\hbar$ and $\omega_{JRF} = 2eV_{RF}/\hbar$.

Substituting V and φ in $I = I_J \sin\varphi + V/R$, we find after some manipulations, using well known harmonic expansions*

$$I = I_J \sum_{\nu=-\infty}^{\infty} J_\nu\left(\frac{\omega_{JRF}}{\omega_{RF}}\right) \sin\left[\varphi_0 + (\underbrace{\omega_{J0} + \nu\, \omega_{RF}}_{\text{side band frequencies}})\, t\right]$$

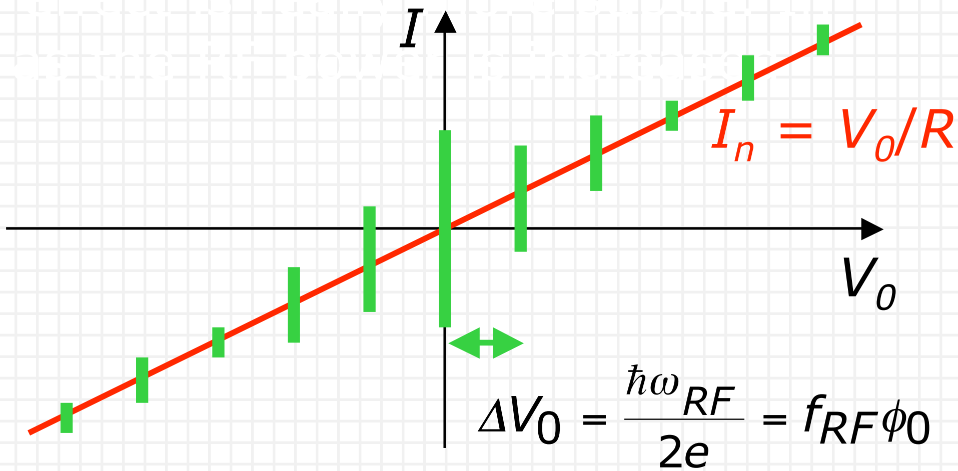
$$+ \frac{V_0}{R} + \frac{V_{RF}}{R} \cos \omega_{RF}\, t$$

$^*\sin(\alpha \sin x) = \sum_{\nu=-\infty\ odd}^{\infty} J_\nu(\alpha)\sin(\nu x)$, $\cos(\alpha \sin x) = \sum_{\nu=-\infty\ \text{even}}^{\infty} J_\nu(\alpha)\cos(\nu x)$

Shapiro Spikes

The AC voltage generates a current response at the sideband frequencies $\omega_{J0} + \nu\omega_{RF}$. The DC part of the current is just V_0/R unless the the Josephson frequency matches a multiple of the AC frequency, $\omega_{J0} + \nu\omega_{RF} = 0$. In that case we generate a DC supercurrent $|I_s| = I_J\, J_\nu(\omega_{JRF}/\omega_{RF})$. This is known as the inverse AC Josephson effect.

In the DC I-V characteristic the supercurrent appears at the so-called Shapiro spikes as shown right.



$$\Delta V_0 = \frac{\hbar\omega_{RF}}{2e} = f_{RF}\phi_0$$

ultra sharp spikes (parts in 10^9)

Gauge Invariant Phase

So far we have ignored the effect of the coupling of the change to the vector potential. This coupling requires that we look for a gauge invariant form of the phase φ. Recall that to obtain a gauge invariant current we required (lecture 2)

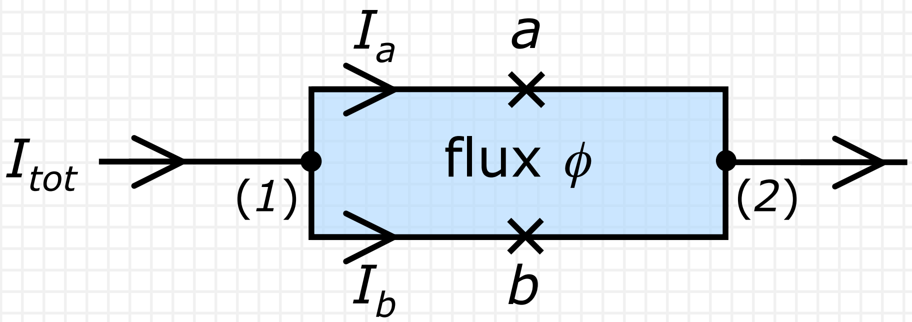
$$\nabla\theta \rightarrow \nabla\theta + \frac{2eA}{\hbar}$$

By integration we arrive at a gauge invariant generalization of the phase φ

$$\varphi = (\theta_1 - \theta_2) \rightarrow (\theta_1 - \theta_2) - \left(\frac{2e}{\hbar}\right) \int_1^2 A \cdot ds$$

This has major consequences for a wide weak link and for two weak links in an applied field. Here we consider two weak links used in the design of a SQUID.

Macroscopic Quantum Interference Between Two Weak Links: Matter Field Interferometer



- Phase change φ_{12} from (1) to (2) is given in two ways

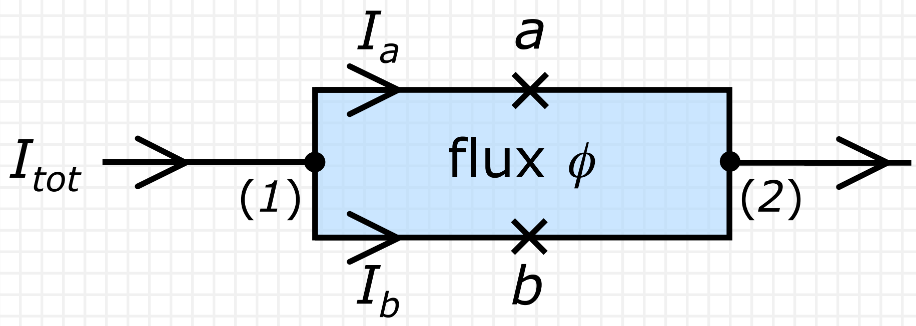
$$\varphi_{12}\,(\text{path } a) = \varphi_a - \frac{2e}{\hbar}\int_{\text{path } a} A \cdot ds$$

$$\varphi_{12}\,(\text{path } b) = \varphi_b - \frac{2e}{\hbar}\int_{\text{path } b} A \cdot ds$$

φ_a is phase change across junction *a*, & φ_b is phase change across junction *b*

- Since φ_{12} (path *a*) = φ_{12} (path *b*), we get

$$\varphi_a - \varphi_b = -\frac{2e}{\hbar}\oint A \cdot ds = -\frac{2e}{\hbar}\phi = -2\pi\frac{\phi}{\phi_0}$$



- Define $\varphi_a + \varphi_b = 2\varphi_{ave}$, so that

$$\varphi_a = \varphi_{ave} - \frac{e}{\hbar}\phi \qquad \varphi_b = \varphi_{ave} + \frac{e}{\hbar}\phi$$

- The total current $I_a + I_b$ is then

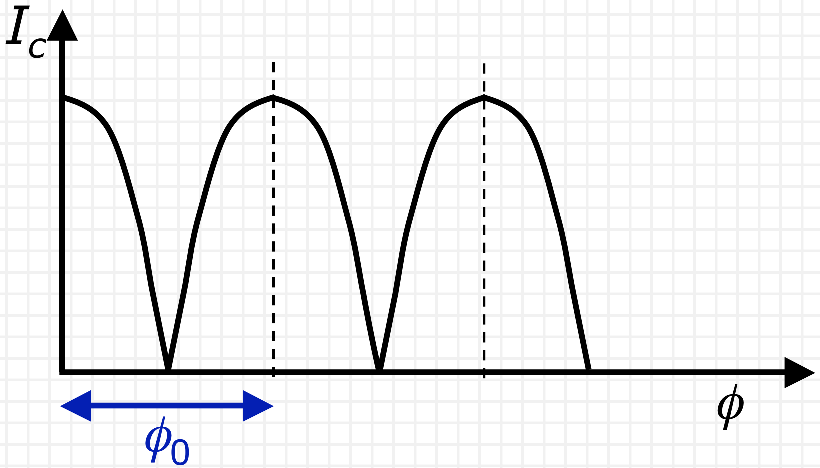
$$I_{tot} = I_J \sin(\varphi_{ave} - \frac{e}{\hbar}\phi) + I_J \sin(\varphi_{ave} + \frac{e}{\hbar}\phi)$$

$$= 2I_J \cos\left(\pi \frac{\phi}{\phi_0}\right) \sin \varphi_{ave}$$

- The critical Josephson current for the pair of links is

$$I_c = 2I_J \left| \cos\left(\pi \frac{\phi}{\phi_0}\right) \right|$$

- I_c oscillates with ϕ with period equal to the flux quantum ϕ_0.

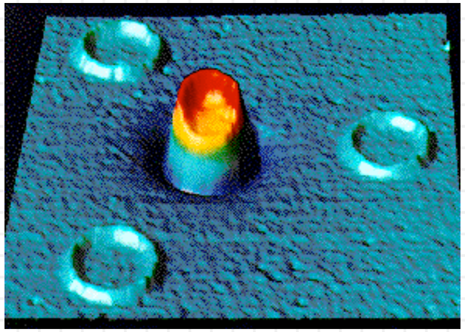


- Analogy to interference from a pair of Young slits, but now for matter waves instead of light waves.

- Superconducting Quantum Interference Device or SQUID: high-sensitivity measurements of magnetic fields, voltages and currents in the *fT*, *fV* and *fA* ranges, respectively.

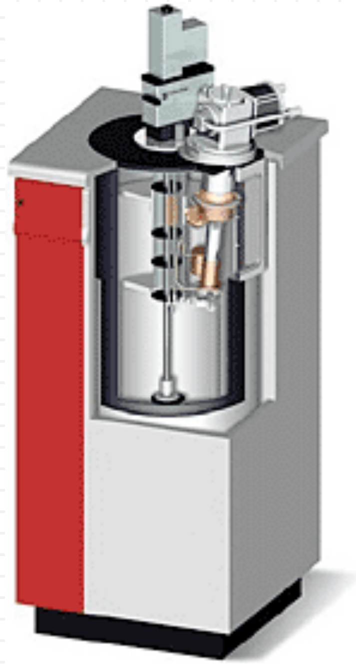
SQUID Applications

The device is highly sensitive: under ideal conditions one can measure a change of $10^{-6}\ \phi_0/\sqrt{\text{Hz}}$. SQUIDs are used as precision magnetometers in the examples below:



Scanning SQUID microscopy
for exotic experiments on high-T_cs (more later…)

Magneto-encephalography
measures tiny magnetic fields (fT range) created by active areas in the brain



Magnetic properties measurement system
for susceptibility measurements etc. – can detect moments down to $\sim 10^{-13}$ Am^2

Picture credits: J. R. Kirtley; 4-D Neuroimaging; Quantum Design

Other Applications of the Josephson Effect

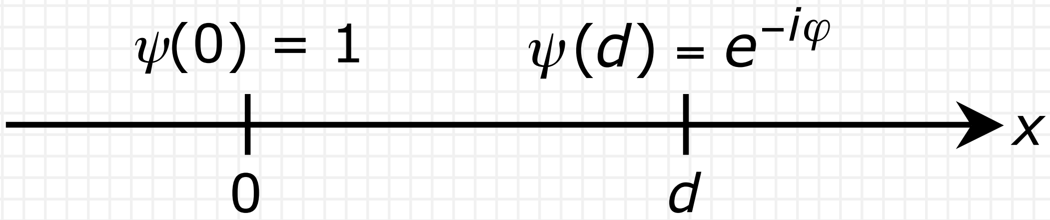
The exactness of the Josephson frequency-voltage relation $\omega_J = 2eV/\hbar$ has led to the adoption of Josephson junction arrays as the primary voltage standard: an incident RF field is tuned to match the Shapiro steps of the array. Frequency can be measured highly accurately, and the Shapiro steps are extremely sharp, giving a relative voltage uncertainty of 1 part in 10^9.

This is one out of several quantum standards that have revolutionized metrology, the quantum Hall effect resistance standard being another prominent example.

Some other applications include microwave detectors and frequency mixers – exploiting the strong nonlinearity and the sideband generation of the weak link, respectively. A lot of research effort is presently going into developing super-conducting transistors and quantum computers using superconducting qubits based on circulating currents and enclosed flux in weak link circuits or non-analytic anyons in exotic pairing states (more later …).

Appendix 1: Short Quasi-1D Superconducting Weak Link

Assume $\beta = -\alpha$:



- for a weak link, $0 < x < d << \xi$, the first GL equation

$$\xi^2 \psi'' + \psi(1 - |\psi|^2) = 0$$

reduces to $\psi'' \cong 0 \rightarrow \psi = a + bx$ so that from boundary

conditions $\psi = 1 - \dfrac{x}{d}(1 - e^{-i\varphi})$

The current is proportional to $-\operatorname{Im}(\psi^* \psi') = \dfrac{\sin\varphi}{d}$

- For a strong link, $0 < x < d >> \xi$, the first GL equation gives

$$\psi\,(1 - |\psi|^2) \cong 0 \rightarrow \psi = \exp(-i\varphi x/d)$$

so that $-\operatorname{Im}(\psi^* \psi') = \dfrac{\varphi}{d}$ (instead of $\dfrac{\sin\varphi}{d}$)

Appendix 2: Simplified Treatment of Combined DC and AC Applied Voltages

- $$I_S = I_J \sin[\underbrace{\omega_{J0} t}_{A} + \underbrace{\frac{\omega_{JRF}}{\omega_{RF}} \sin(\omega_{RF} t)}_{B}] \quad \text{if} \quad \varphi_0 = 0$$

  $$= I_J[\sin A \cos B + \cos A \underbrace{\sin B}_{B + \dots}]$$

- The term (cos *A*) *B* is proportional to

  $$\cos(\omega_{J0} t)\sin(\omega_{RF} t)$$

  $$= \frac{1}{2}\sin[(\omega_{RF} + \omega_{J0})t] + \frac{1}{2}\sin[\underbrace{(\omega_{RF} - \omega_{J0})}_{\text{becomes DC if } \omega_{RF} = \omega_{J0} = \frac{2eV}{\hbar}} t]$$

- Thus, we will get a DC Josephson effect when

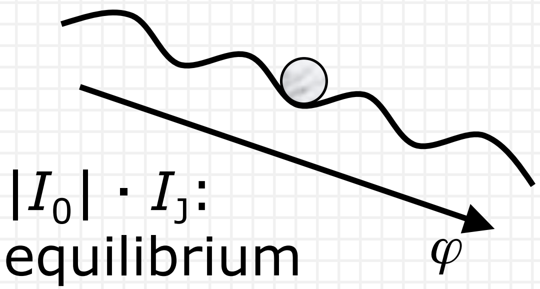
  $$V = \frac{\hbar}{2e}\omega_{RF}$$

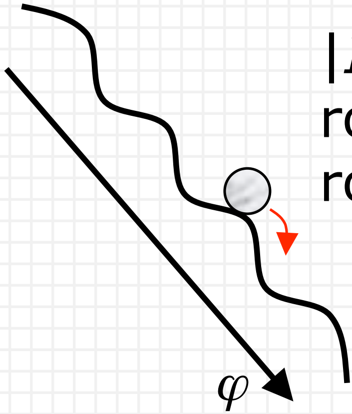
Appendix 3: Current Biased Weak Links

In practice, it is usually the current rather than the voltage that is controlled in a weak link. We need to invert the Josephson phase equation. Suppose first that we apply a DC current I_0 to the RSJ. Then from p. 5 the phase φ is given by

$$\frac{\hbar}{2eR}\frac{\partial \varphi}{\partial t} = I_0 - I_J \sin \varphi$$

If $|I_0| \le I_J$, the phase reaches an equilibrium value given by $\partial \varphi / \partial t = 0$, i.e., $I_0 = I_J \sin \varphi_{equil}$. Note that $\partial \varphi / \partial t = 0$ means $V = 0$. If $|I_0| > I_J$ such an equilibrium is not possible and φ keeps changing with time.





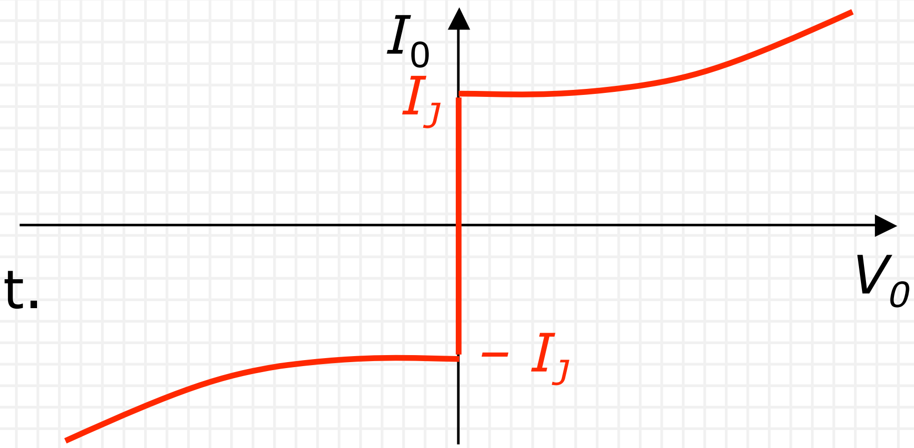
It can be shown that for a given I_0 the phase φ satisfies

$$I_0 \tan\frac{\varphi}{2} = I_J + \frac{V_0}{R}\tan\frac{\omega_J t}{2}$$

where V_0 is the mean voltage across the link defined by

$$I_0 = \sqrt{I_J^2 + V_0^2 / R^2}$$

This gives the current-bias I-V characteristic shown on the right.



The I-V characteristic under current bias and under voltage bias are therefore quite different.

Generally, the weak link equations have to be integrated numerically.