Last Lecture (#1):

The basic superconducting state is characterised by

$$\rho = 0 \qquad B = 0^{+}$$

and by some internal order (yet to be fully identified).

$^{+}$The condition $B = 0$ ceases to be valid at high fields in so-called type-II superconductors where vortex lines enter the superconductor (lecture 3).

We introduced the phenomenological Ginzburg-Landau theory based on a complex-valued order parameter. The theory will be used to describe the above properties of the superconducting state and, in particular, the spatial variation of the order parameter near surfaces or in a magnetic field.

Lecture 2:

- Critical field B_c
- Ginzburg-Landau equations – what happens if we try to minimise the Ginzburg-Landau free energy?
- London equations – field expulsion and resistance loss
- Penetration depth and coherence length as the two characteristic length scales of superconductivity
- Gauge transformations and U(1) symmetry breaking

- Literature: Waldram chs. 2 & 4
(or equivalent chapters in Annett, Schmidt, or Tinkham)

The Critical Field B_c

Recall the Ginzburg-Landau (GL) free energy density

$$f = \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m}\left|-i\hbar\nabla\psi + 2eA\psi\right|^2 + \frac{(B - B_E)^2}{2\mu_0}$$

First we look for homogeneous solutions for ψ deep inside the bulk of the superconductor (sc). Consider the two cases:

normal state ($\alpha > 0$): $\quad |\psi_0| = 0, \quad B = B_E, \quad f = 0$

sc state ($\alpha < 0$): $\quad |\psi_0| = \sqrt{\frac{-\alpha}{\beta}} \neq 0, \quad B = 0, \quad f = -\frac{\alpha^2}{2\beta} + \frac{B_E^2}{2\mu_0}$

The sc state is thus evidently stable ($f < 0$) below a critical field

$$B_c \equiv \sqrt{\frac{\mu_0 \alpha^2}{\beta}}$$

Minimise GL Free Energy

Close to surfaces and in the presence of vortices (next lecture) where the magnetic field is incompletely screened and the supercurrent density is finite we need to look for spatially inhomogeneous solutions.

To find the most probable configurations of the fields $\psi(r)$ and $A(r)$, we minimise the free energy density w.r.t. these fields that in general can vary in a complicated way with position in the superconductor.

We use the usual shortcut of taking the derivatives w.r.t. ψ and ψ^* rather than Re ψ and Im ψ, but the end result is of course the same. The parameters of the GL model other than the ψ and A fields are held constant. The most probable configuration will be assumed to be the equilibrium average configuration.

(i) The functional derivative w.r.t. ψ: To see what key mathematical steps are needed, consider first the simpler free energy density for a real order parameter ψ

$$f = \alpha\psi^2 + \frac{\beta}{2}\psi^4 + \gamma|\nabla\psi|^2$$

$$\delta f = 2\alpha\psi\delta\psi + 2\beta\psi^3\delta\psi + \gamma\delta|\nabla\psi|^2$$

The problem is how to handle the last term: to first order in $\delta\psi$

$$\delta|\nabla\psi|^2 = |\nabla(\psi + \delta\psi)|^2 - |\nabla\psi|^2 = 2\nabla\psi\cdot\nabla\delta\psi$$

Next note that

$$\underbrace{\nabla\cdot((\nabla\psi)\delta\psi)}_{\text{gives surface contribution}} = (\nabla^2\psi)\delta\psi + \nabla\psi\cdot\nabla\delta\psi$$

Thus, in the bulk

$$\delta f = 2\delta\psi[\underbrace{-\gamma\nabla^2\psi + (\alpha + \beta\psi^2)\psi}_{=0}] = 0$$

if $\delta f = 0$ for any variation $\delta\psi$

Similarly, the functional derivative of the GL free energy density w.r.t ψ* gives:

$$\frac{1}{2m}(-i\hbar\nabla + 2eA)^2\psi + (\alpha + \beta|\psi|^2)\psi = 0$$

First Ginzburg-Landau equation

(ii) The functional derivative w.r.t. A : this requires manipulations similar to that above, but now with

$$\underbrace{\nabla \times (\underbrace{\nabla \times A}_{B} \cdot \delta A)}_{\text{gives surface contribution}} = \underbrace{\nabla \times B}_{\mu_0 J} \cdot \delta A + B \cdot \nabla \times \delta A$$

Thus, in the bulk we find:

setting $\psi = n_s^{1/2}\exp(i\theta)$

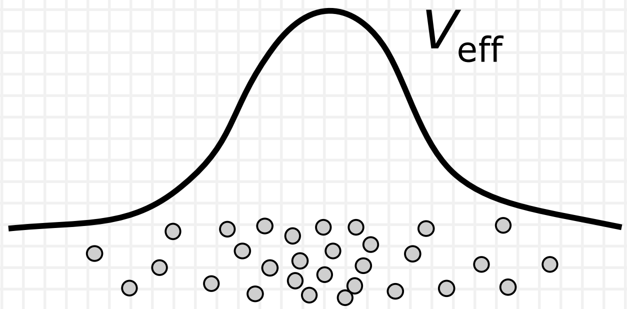
$$J_s = \frac{ie\hbar}{m}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{4e^2}{m}A\psi^*\psi \quad = \frac{-2e\hbar n_s}{m}\left(\nabla\theta + \frac{2eA}{\hbar}\right)$$

Second Ginzburg-Landau equation

Comments on GL Equations

The first GL equation has the form of the free-particle Schrödinger equation – were it not for the non-linear ($\psi^*\psi\psi$) term. This means we expect ψ to behave in many respects like a macroscopic wavefunction, but without certain properties associated with linearity: superposition and normalisation. Also, *A* now includes the screening field, which yields B=0 deep inside the sc (next slide).

Without the *A*-term it is also known as the Gross-Pitaevskii equation in the context of Bose-Einstein condensates. It can be inferred empirically as arising from an effective potential $V_{\text{eff}} \propto \psi^*\psi$ (lecture 7)

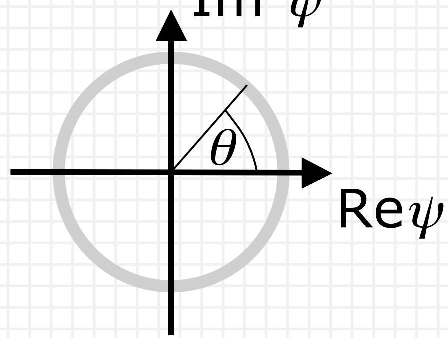


The second GL equation is identical to the usual quantum-mechanical definition of current.

The GL equations contain the essentials of superconductivity: we consider in this lecture the Meissner effect and zero resistance.

Meissner Effect

Let's make the London approximation and assume that the amplitude $|\psi|$ is fixed near the equilibrium value $|\psi|^2 = n_s = -\alpha/\beta$ and that $\psi = |\psi| exp(i\theta)$.



Then from the curl of the 2nd GL equation:

$$\text{curl } J_s = \frac{-4e^2 n_s}{m} B \quad \text{(since curl grad}\theta = 0 \text{ and curl } A = B)$$

also

$$\mu_0 \text{curl } J_s = \text{curl curl } B = (\text{grad} \underbrace{\text{div } B}_{0} - \nabla^2 B) \quad (\text{since } \mu_0 J_s = \text{curl} B)$$

Thus,

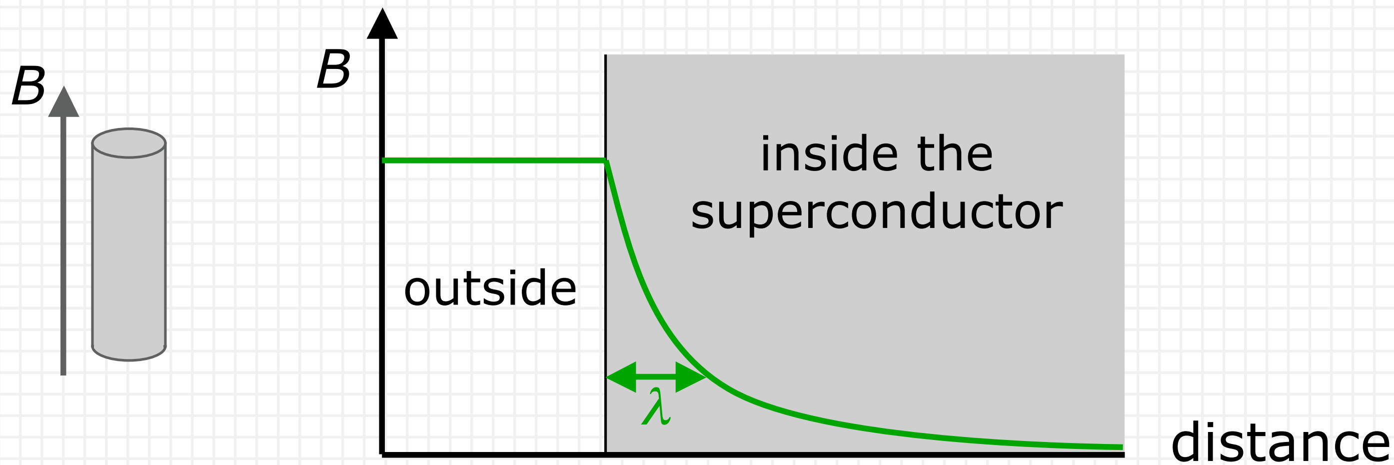
$$\nabla^2 B = \frac{B}{\lambda^2} \qquad \lambda = \sqrt{\frac{m}{4\mu_0 e^2 n_s}}$$

2nd London Equation

(see below, p. 13, for 1st London equation)

Penetration Depth

The last equation describes magnetic screening: B decays exponentially inside a superconductor, with characteristic length scale, λ , known as the penetration depth. λ is typically of the order of 1000 Å (examples sheet).



Note that the value of λ depends only on the superfluid density n_s and on the effective mass m. As $T \rightarrow 0$, $n_s \rightarrow n_e/2$ so that in this limit λ is purely determined by normal state properties. At T_c , $n_s \rightarrow 0$, so that λ diverges and screening breaks down.

Gauge Transformations

Thus, the GL theory describes the Meissner effect. What about zero resistance itself? Obviously, the screening currents associated with the Meissner effect are persistent and must therefore flow without resistance. To make a general case for zero resistance and for later reference it is instructive to reconsider the problem of gauge transformations.

Recall the forms of the electric and magnetic fields in terms of the scalar and vector potentials ϕ and A:

$$E = -\nabla\phi - \partial A / \partial t \qquad B = \text{curl } A$$

We can easily see that the observable fields E and B remain unchanged under the gauge transformation

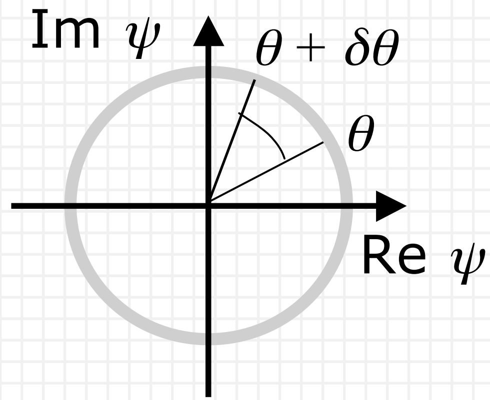
$$A \rightarrow A + \nabla\chi \qquad \phi \rightarrow \phi - \partial\chi / \partial t$$

where $\chi(r,t)$ is an arbitrary differentiable and single-valued function.

The supercurrent $$J_s = \frac{-2e\hbar n_s}{m}\left(\nabla\theta + \frac{2eA}{\hbar}\right)$$

which is an observable, remains unchanged if the phase θ of the sc wavefunction simultaneously transforms as

$$\boxed{\theta \rightarrow \theta - \frac{2e}{\hbar}\chi} = \theta + \delta\theta$$



Thus, a gauge transformation evidently corresponds to a rotation in the internal space of the matter field at each point (r,t).

Time Dependence of Phase

The time dependence of ψ cannot be inferred from the GL theory and is in general complicated. However, London suggested that the time derivative of θ might be expected to have a contribution of the form

$$\boxed{\frac{\partial \theta}{\partial t} = \frac{2e}{\hbar}\phi}$$

a result familiar from quantum mechanics, and consistent with the gauge transformations

$$\frac{\partial \theta}{\partial t} \to \frac{\partial \theta}{\partial t} - \frac{2e}{\hbar}\frac{\partial \chi}{\partial t} \quad \text{and} \quad \phi \to \phi - \frac{\partial \chi}{\partial t}$$

The relations between J, θ and ϕ allows us to write

$$\frac{\partial J_s}{\partial t} = -\frac{2e\hbar n_s}{m}\left(\frac{2e}{\hbar}\nabla\phi + \frac{2e}{\hbar}\frac{\partial A}{\partial t}\right)$$

and, from $E = -\nabla\phi - \frac{\partial A}{\partial t}$, thus obtain

$$\frac{\partial J_s}{\partial t} = \frac{4e^2 n_s}{m} E$$

1st London equation

(see p. 8, for 2nd London equation)

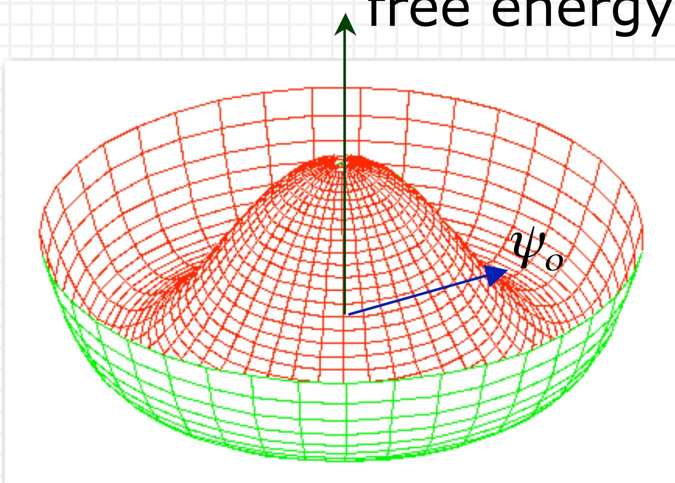
This is an acceleration equation – an electric field is only needed to kick-start a supercurrent, but not to sustain it.

Historically, the two London equations predate the GL treatment, and both of course predate the microscopic BCS theory.

Gauge Symmetry Breaking

J, ϕ and *A* are affected by χ only through its time and space derivatives. Thus, we have the freedom to add a global constant, $\delta\theta = -2e\chi/\hbar$, to the phase θ of ψ without changing the the current and the potentials – a property known as global gauge symmetry. As this corresponds to a multiplication of ψ with exp($i\delta\theta$) and since these c-numbers form the multiplicative group U(1) this symmetry is referred to as U(1) (gauge) symmetry.

A particular choice of $\psi = \psi_0$ in the GL theory breaks global (not local) U(1) symmetry. This is visualised by means of the "mexican hat" (or "bottom of wine bottle") potential shown right. (Qualifications will be given in lecture 12.)



Broken global U(1) symmetry

Coherence Length

The GL free energy has already provided us with one characteristic length scale, the penetration depth λ, which is the scale over which magnetic fields vary in a sc.

From the non-linear Schrödinger equation (p. 6 with $\alpha = -|\alpha|$), we can define a second length scale, the GL coherence length ξ,

$$\xi^2\nabla^2\psi + \psi - \frac{\beta}{|\alpha|}|\psi|^2\psi = 0 \qquad \xi = \sqrt{\frac{\hbar^2}{2m|\alpha|}}$$

This gives the scale over which $|\psi|$ itself will vary.

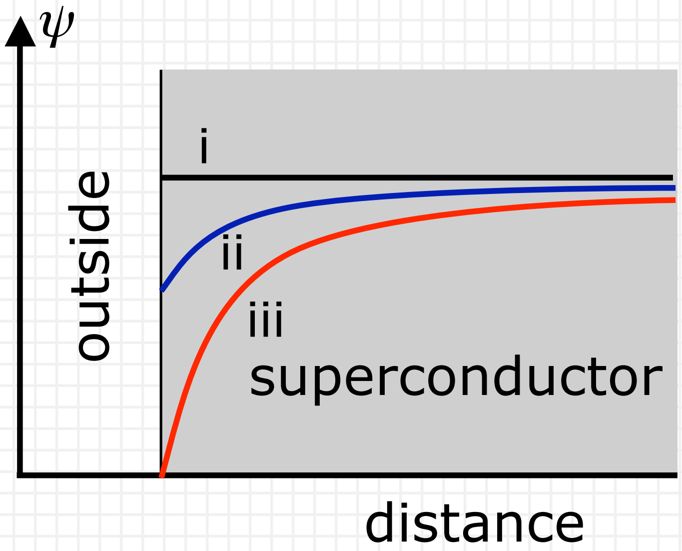
Unlike λ, the coherence length ξ cannot easily be related to normal state properties since it depends crucially on α. ξ diverges at T_c and $T \to 0$ values can vary greatly from material to material (10 Å to 10,000 Å).

Appendix: Boundary Conditions

(i) SC – vacuum interface:

$$(-i\hbar\nabla_n + 2eA_n)\psi = 0$$

n means normal component



This implies that there is no current flow across the surface and that the full value of ψ continues right up to the surface (a thin film will normally have the same T_c as the bulk).

(ii) SC – normal metal interface:

$$(-i\hbar\nabla_n + 2eA_n)\psi = \frac{-i\hbar\psi}{\ell},$$ where ℓ is a characteristic length over which superconductivity is induced in the normal metal.

(iii) SC – ferromagnet interface:

$\psi = 0$ at the interface

for usual case where ferromagnetism destroys superconductivity.

(see e.g., Waldram pp 45-46)