Superconductivity & Quantum Coherence **Lent 2012**
Examples Sheet 3

1. Bogoliubov Transformation

In the lectures, we encountered a Hamiltonian of the form $\hat{H} = \varepsilon_0(a^+a + b^+b) + \varepsilon_1(a^+b^+ + ab)$, where a^+/a and b^+/b each create/annihilate a bosonic state. We want to rewrite this Hamiltonian in diagonal bosonic form $\hat{H} = \varepsilon_{GS} + \varepsilon_B(\alpha^+\alpha + \beta^+\beta)$.

(a) We try to write α, β, $\alpha+$, $\beta+$ as linear combinations of a, b, $a+$, $b+$, as in $\alpha = ua + vb^+$ and $\beta = ub + va^+$, with real coefficients u and v. Show that α, β, $\alpha+$, $\beta+$ are bosonic operators if and only if $u^2 - v^2 = 1$.
(b) Express $\hat{H}$ in terms of these new operators and show that the "anomalous" terms $\alpha\beta$, $\alpha+\beta+$ vanish if $\varepsilon_1(u^2 + v^2) = 2uv\varepsilon_0$.
(c) From these two conditions determine u and v, as well as ε_{GS} and $\varepsilon_{\mathbf{B}}$.

2. Condensate Depletion

The particle number operator is $\hat{N} = \sum_k a_k^+ a_k$.

(a) Rewrite $\hat{N}$ in terms of the Bogoliubov bosonic operators α_k, β_k, α_k^+, β_k^+, as in the first question on this sheet, and show that its expectation value in the ground state is given by $\langle \hat{N} \rangle = N_0 + \sum_{k \neq 0} v_k^2$.
(b) Convert the sum to an integral in momentum space, and show that the condensate depletion, i.e., the particle density in the one-particle states with $k \neq 0$ when the overall system is in the many-body ground state, is given by

$$n - n_0 = \frac{1}{3\pi^2}\left(\frac{mc}{\hbar}\right)^3$$

where $n = \langle \hat{N} \rangle / V$, $n_0 = N_0 / V$ and V is the volume.

(c) From this formula, what value of $(n - n_0)$ do we expect in superfluid ^{4}He? Comment on your results.

[The first two questions are covered in Pethick & Smith, pp. 207-211.
Answer for (c): $\sim 1 \times 10^{29}$ m^{-3}, i.e., about five times the overall particle density.]

3. Weakly Bound States and Cooper Pairs

Show that a weak, short range attractive potential between two particles always leads to a bound state in 1D and 2D, but not in 3D.

Also show that in 3D a weakly bound state *does* exist if the particles are fermions that, in the centre-of-mass frame, have wavevectors just outside a filled Fermi sphere of free electrons. For this case, assume that the interaction matrix element is BCS-like, i.e., it's a constant $g = -|g|$ as long as $\varepsilon_k, \varepsilon_{k'} < \varepsilon_F + \varepsilon_c$.

[Hint: For the first part, write down the Schrödinger equation for the relative motion of the particles and take the Fourier transform φ_k of the wavefunction $\varphi(r)$. The wavefunction is strongly peaked in Fourier space (why?), so the Fourier transform V_k of the potential energy $V(r)$ can be taken as a constant $-|g|$ over the range where φ_k is appreciable. Self-consistency for the value of $C \equiv \int \varphi_k \, dk$ gives the binding energy. The second part is conceptually identical - except that φ_k is now peaked on a thin shell around the Fermi sphere. If you get stuck on the second part, have a look at Annett, pp. 131-133.]

4. Temperature Dependence of the BCS Gap

Starting with the Bogoliubov solution for the BCS model, derive the expression for the temperature dependence of the BCS energy gap $\Delta(T)$ given in lecture 10.

5. Gap Nodes

Write a brief essay about how one can establish the existence and location of gap nodes through experiments.

6. d-Wave Josephson Junction

Write a brief essay on d-wave superconductivity and d-wave Josephson junctions.