Superconductivity & Quantum Coherence — Lent 2012
Examples Sheet 1

1. Superconductivity and Resistivity in the Elements

It is sometimes stated that "superconductivity is more likely to be observed in metals whose normal state is highly resistive" (e.g., Waldram, p. 118). The reasoning is that the high temperature normal state resistivity and the BCS superconducting transition temperature are both enhanced by strong electron-phonon coupling.

Try to check this statement for the elemental superconductors, using the available data on www.webelements.com, by producing a scatter plot of T_c vs the room temperature resistivity. Comment on your results.

2. Levitating Magnet

(a) Show that magnetic field lines outside a superconductor are tangential to the surface.

(b) A small permanent magnet with dipole moment m and mass M is placed above the horizontal surface of a superconductor. Show that in its equilibrium position the magnet levitates at a height

$$h = \left(\frac{3\mu_0 m^2}{64\pi M g} \right)^{1/4}$$

above the surface, with its dipole moment pointing parallel to surface.
[Hint: use mirror dipoles, just as you can use mirror charges in electrostatic problems.]

(c) Plug in the numbers for small but powerful NdFeB magnet (volume $V = 30$ mm^3, magnetisation density $\mu_0 \mathcal{M} \cong 1$ T, density $\rho \cong 8$ g/cm^3). [Answer: 8.2 mm]

3. The Landau Model

In the Landau model the free energy as a function of the order parameter ψ and the temperature T is of the form

$$f = a(T - T_c)\psi^2 + \frac{\beta}{2}\psi^4$$

Determine and sketch the temperature dependence of the equilibrium value of the order parameter $\psi_0(T)$, the equilibrium free energy density $f_0(T)$, and the specific heat $C(T)$. In particular, show that the model leads to a specific heat anomaly at T_c in the form of a discontinuity of height $\Delta C = a^2 T_c/\beta$.

4. Field Penetration and Screening Currents

A type-I superconducting foil of thickness a and with penetration depth λ is exposed to a homogeneous external magnetic field $B_E < B_c$ parallel to the surface of the foil. Let's assume that the foil carries no net current, but the field of course induces screening supercurrents near the surface.

(a) Calculate the field distribution $B(z)$, and the supercurrent density $J_s(z)$, along the normal $\hat{u}_z$ of the foil $(-a/2 < z < a/2)$.

(b) Determine the vector potential $\vec{A}(\vec{r},t)$ in the so-called rigid gauge where the phase of the wavefunction is fixed at $\theta = 0$ everywhere and at all times t.

(b) Show explicitly that $\nabla \cdot \vec{A} = 0$. The gauge with this property is called the London gauge. Discuss under what conditions London and rigid gauges are equivalent.

(c) Determine both the vector potential $\vec{A}(\vec{r},t)$ and the electric potential $\phi(\vec{r},t)$ if we make a gauge transformation with $\chi(\vec{r},t) = (\hbar/2e)(\vec{k}\cdot\vec{r} - \omega t)$ for some frequency ω and wavevector $\vec{k}$.

5. The $B - T$ Phase Diagram

Write an essay about the behaviour of type-I and type-II superconductors in applied magnetic fields. In particular, distinguish between the critical fields B_c, B_{c1} and B_{c2}; discuss the order of the phase transitions and summarise the behaviour in appropriate $B - T$ phase diagrams. [A qualitative discussion is enough — no need to include the algebra here.]

6. Magnetic Field Inhomogeneity in the Vortex State

Sketch the currents inside a type-II superconducting cylinder for an applied field B_E greater than B_{c1}, but lower than B_{c2} and pointing along the cylinder axis. Include the screening currents round the outside of the cylinder, and indicate current and field directions.

Determine the average field B in the vortex state of Nb_3Sn in an applied field of 1 T, at low temperatures. Estimate the magnitude of the spread of this field. [The values for ξ and λ were given in the lecture. Use approximations as appropriate. Answer: 0.981 ± 0.019 T.]

7. Vortex Creep

(a) A superconducting wire of length $l = 8$ cm is placed in an external field, $B_{c1} << B_E = 5$ T, perpendicular to the wire. The superconductor is in the resistive (vortex) state and the voltage across the wire is $U = 1.5 \times 10^{-5}$ V. Find the velocity v_L of the moving vortices.

(b) If the vortex lattice order is not destroyed by the movement, the time periodicity of the "vortex procession" should lead to AC electromagnetic radiation. Determine the expected characteristic radiation frequency $2\pi\omega$ under the simplifying assumption that the vortex lattice is square, with one axis along the wire. [Answers: 3.8×10^{-5} m/s and 1.9 kHz. Alas, this radiation is "very difficult to observe" (Schmidt, p. 135).]